$$-1 < \theta < 1 \quad \Rightarrow \quad \sum_{k=0}^{\infty} \theta^k = \frac{1}{1-\theta} \qquad \sum_{k=1}^{\infty} \theta^k = \frac{\theta}{1-\theta} \qquad \frac{d}{dx}\log x = \frac{1}{x}$$

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \text{Binomial Thm } (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\text{Cov}(X,Y) = E[(X-EX)(Y-EY)] = E(XY) - E(X)E(Y)$$

$$\gamma(h) = \text{Cov}(X_{t+h}, X_t) = E(X_{t+h}X_t) \text{ when } EX = 0 \qquad \gamma(0) = \sigma_X^2 = \text{Var}\, X = \text{Cov}(X,X)$$

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h}-\bar{x})(x_t-\bar{x}) \qquad \nabla_{12}X_t = X_t - X_{t-12} \qquad \nabla^d = (1-B)^d$$

$$\cos A \cos B + \sin A \sin B = \cos(A-B) = \cos(B-A)$$

$$\cos 2\lambda = 2\cos^2\lambda - 1 \qquad 2\cos\theta = e^{i\theta} + e^{-i\theta} \qquad 2i\sin\theta = e^{i\theta} - e^{-i\theta} \qquad (-1)^{|h|} = \cos(\pi h)$$

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \qquad aX + b \sim N(a\mu_X + b, a^2\sigma_X^2)$$

Estimation of μ

$$\sqrt{n}\,(\bar{X} - \mu) \sim N\left(0, \sum_{h=-\infty}^{\infty} \gamma(h)\right) \quad \Rightarrow \quad \bar{X} \sim N\left(\mu, \frac{1}{n}\sum_{h=-\infty}^{\infty}\gamma(h)\right)$$

The 95% confidence interval for μ is $\bar{x} \pm 1.96\sqrt{v/n}$ where $v = \sum_{h=-\infty}^{\infty}\gamma(h)$

$$\text{AR(1)} \quad \Rightarrow \quad v = \frac{\sigma^2}{(1-\phi)^2} \qquad \text{MA(1)} \quad \Rightarrow \quad v = \sigma^2(1 + 2\theta + \theta^2)$$

Best Linear Predictor

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} \qquad a_0 = \mu(1 - a_1 - a_2)$$

$$P_n X_{n+h} = \mu + a_1(X_n - \mu) + \cdots + a_n(X_1 - \mu) = a_0 + a_1 X_n + \cdots + a_n X_1$$

$$E(X_{n+h} - P_n X_{n+h})^2 = \gamma(0) - \mathbf{a}_n^T \gamma_n(h)$$

$$E(X_{n+h} - P_n X_{n+h}) = 0$$

$$E[(X_{n+h} - P_n X_{n+h})X_j] = 0,\ j = 1, \ldots, n$$

$$E(X_{n+1} - \hat{X}_{n+1})^2 = \sigma^2 r_n = v_n$$

Innovations $\quad v_0 = E[(X_1 - \hat{X}_1)^2] = \gamma(0) \qquad \theta_{11} = \dfrac{\gamma(1)}{\gamma(0)}$

$$v_1 = E[(X_2 - \hat{X}_2)^2] = \gamma(0) - \frac{\gamma(1)^2}{\gamma(0)} \qquad \theta_{22} = \frac{\gamma(2)}{\gamma(0)} \qquad \theta_{21} = \frac{\gamma(1)[\gamma(0) - \gamma(2)]}{\gamma(0)^2 - \gamma(1)^2} \qquad \theta_{33} = \frac{\gamma(3)}{\gamma(0)}$$

MA(∞) $\quad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ $\quad$ AR(∞) $\quad Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$

AR(1) $\quad X_t - \phi X_{t-1} = Z_t \quad \{Z_t\} \sim \text{WN}(0, \sigma^2) \quad \gamma_X(h) = \frac{\sigma^2 \phi^{|h|}}{1-\phi^2} \quad \rho_X(h) = \phi^{|h|}$

$$\hat{\phi} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \quad \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\gamma}(1)\hat{\phi} \quad \bar{x} \pm 1.96 \frac{\sigma}{(1-\phi)\sqrt{n}} \quad E(X_{n+h} - P_n X_{n+h})^2 = \frac{\sigma^2(1-\phi^{2h})}{1-\phi^2}$$

$$|\phi| < 1 \Rightarrow \quad X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} \quad |\phi| > 1 \Rightarrow \quad X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j} \quad \sum_{h=-\infty}^{\infty} \gamma_X(h) = \frac{\sigma^2}{(1-\phi)^2}$$

$$r_0 = \frac{1}{1-\phi^2} \quad r_j = 1 \text{ for } j \geq 1 \quad f(\lambda) = \frac{\sigma^2}{2\pi} \left(1 - 2\phi \cos\lambda + \phi^2\right)^{-1}$$

MA(1) $\quad X_t = Z_t + \theta Z_{t-1} \quad \{Z_t\} \sim \text{WN}(0, \sigma^2) \quad \gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & h = 0 \\ \sigma^2\theta & h = \pm 1 \\ 0 & |h| > 1 \end{cases}$

$$\bar{x}_n \pm 1.96\sqrt{v/n} \quad v = \sum_{h=-\infty}^{\infty} \gamma(h) = \sigma^2(1 + 2\theta + \theta^2) \quad f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + 2\theta\cos\lambda + \theta^2\right)$$

ARMA(1,1)

$\psi_0 = 1 \quad \psi_j = (\phi + \theta)\phi^{j-1},\ j = 1, 2, \ldots$

$\pi_0 = 1 \quad \pi_j = -(\phi + \theta)(-\theta)^{j-1},\ j = 1, 2, \ldots$

$$\gamma(0) = \sigma^2 \left[1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right] \quad \gamma(1) = \sigma^2 \left[\theta + \phi + \frac{(\theta+\phi)^2\phi}{1-\phi^2}\right] \quad \gamma(h) = \phi^{h-1}\gamma(1),\ h \geq 2$$

ARMA(p,q)

$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ Roots of $\phi(z)$ greater than 1 implies causality.

$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ Roots of $\theta(z)$ greater than 1 implies invertibility.

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots$$

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k} \quad \pi_j = -\phi_j - \sum_{k=1}^{q} \theta_k \pi_{j-k} \quad f(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|\theta\left(e^{-i\lambda}\right)\right|^2}{\left|\phi\left(e^{-i\lambda}\right)\right|^2} \quad -\pi \leq \lambda \leq \pi$$

Yule-Walker $\quad \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{pmatrix}^{-1} \begin{pmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{pmatrix} \quad \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\gamma}(1)\hat{\phi}_1 - \hat{\gamma}(2)\hat{\phi}_2$

Forecast Function $\quad P_n X_{n+h} = g(h) = a_0 + a_1 h + \cdots + a_{d-1}h^{d-1} + b_1 \xi_1^{-h} + \cdots + b_p \xi_p^{-h}$

$$g(0) = x_n \quad g(-1) = x_{n-1} \quad \phi^*(z) = (1-z)^d \phi(z) = (1-z)^d (1 - \xi_1^{-1} z) \cdots (1 - \xi_p^{-1} z)$$

3.2.1 Calculation of the ACVF

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},\ \{Z_t\} \sim \mathrm{WN}(0,\sigma^2) \quad \Rightarrow \quad \gamma_X(h) = E(X_{t+h}X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

$$\gamma_X(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \qquad \mathrm{MA}(q) \Rightarrow \quad \gamma_X(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & |h| \le q \\ 0 & |h| > q \end{cases}$$

4.1 Spectral Densities

$$\mathrm{Var}\,\bar{X}_n = \frac{1}{n} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma(h) \approx f(0) \qquad \mathrm{WN} \Rightarrow f(\lambda) = \frac{\sigma^2}{2\pi}$$

λ = radians $\quad$ ω = radians/sec $\quad$ period = 2π radians/ω = seconds

$f(\lambda)$ = spectral density function $\quad$ $F(\lambda)$ = spectral distribution function

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{1}{2\pi} \left[\gamma(0) + \sum_{h=1}^{\infty} (e^{-ih\lambda} + e^{ih\lambda}) \gamma(h) \right],\ -\infty < \lambda < \infty$$

$$f(\lambda) \ge 0 \qquad f(-\lambda) = f(\lambda) \qquad \int_{-\pi}^{\pi} f(\lambda)\,d\lambda = \gamma(0) < \infty \qquad f(\lambda + 2\pi) = f(\lambda)$$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\,d\lambda = \int_{-\pi}^{\pi} \cos(h\lambda) f(\lambda)\,d\lambda$$

$$F(\lambda) = \int_{-\pi}^{\lambda} f(y)\,dy \qquad F(-\pi) = 0 \qquad F(\pi) = \gamma(0) \qquad d\,F(\lambda) = f(\lambda)\,d\lambda$$

$$X_t = A\cos(\omega t) + B\sin(\omega t);\ A, B \sim (0, \nu^2);\ \gamma_X(h) = \nu^2 \cos(\omega h);\ F_X(\lambda) = \begin{cases} 0 & \lambda < -\omega \\ \nu^2/2 & -\omega \le \lambda < \omega \\ \nu^2 & \lambda \ge \omega \end{cases}$$

4.3 Time-Invariant Linear Filters

$$\text{Filter } Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} \quad \Rightarrow \quad f_Y(\lambda) = \left|\Psi\left(e^{-i\lambda}\right)\right|^2 f_X(\lambda) \quad = \Psi\left(e^{-i\lambda}\right) \Psi\left(e^{i\lambda}\right) f_X(\lambda)$$

$$\text{Transfer function } \Psi\left(e^{-i\lambda}\right) = \sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda} \qquad \text{Power transfer function } \left|\Psi\left(e^{-i\lambda}\right)\right|^2$$

$$Y_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j} \quad \Rightarrow \quad f_Y(\lambda) = \frac{1}{2q+1} \left(\sum_{j=-q}^{q} e^{-ij\lambda} \right)^2 f_X(\lambda)$$

$$q = 1 \quad \Rightarrow \quad f_Y(\lambda) = \tfrac{1}{3} \left(e^{i\lambda} + 1 + e^{-i\lambda}\right)^2 f_X(\lambda)$$

5.3 Diagnostic Checking

$$\hat{W}_t = \frac{X_t - \hat{X}_t}{\sqrt{r_{t-1}}} \quad t = 1, \ldots, n \qquad \text{The noise band is } \pm 1.96/\sqrt{n}$$

5.4 Forecasting $\quad P_n X_{n+1} = \phi_1 X_n + \cdots + \phi_p X_{n+1-p} + \sum_{j=1}^{q} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) \qquad \theta_{nj} \approx \theta_j$

$$X_{n+h} - \hat{X}_{n+h} = 0 \text{ for } h \geq 1 \qquad v_n = \sigma^2 r_n = E[(X_{n+1} - P_n X_{n+1})^2] = \sigma_n^2(1) \approx \sigma^2 \sum_{j=0}^{0} \psi_j^2 = \sigma^2$$

5.5 Order Selection $\quad \text{AICC} = -2\log L + 2(p+q+1)n/(n-p-q-2)$

$$L = \frac{1}{\sqrt{(2\pi)^n v_0 \cdots v_{n-1}}} \exp\left[-\frac{1}{2}\sum_{j=1}^{n} \frac{(X_j - \hat{X}_j)^2}{v_{j-1}}\right] \qquad v_n = \sigma^2 r_n$$

$$\ell = \log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\sum_{j=1}^{n}\log r_{j-1} - \frac{1}{2\sigma^2}\sum_{j=1}^{n}\frac{(X_j - \hat{X}_j)^2}{r_{j-1}}$$

Solve $\dfrac{\partial \ell}{\partial \sigma^2} = 0$ to obtain $\hat{\sigma}^2 = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\frac{(X_j - \hat{X}_j)^2}{r_{j-1}}$

Maximize reduced log likelihood $\ell^* = -\dfrac{n}{2}\log\hat{\sigma}^2 - \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\log r_{j-1}$

Or multiply by $-2/n$ and minimize $\ell^* = \log\hat{\sigma}^2 + \dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\log r_{j-1}$ $\qquad$ (5.2.12) on p. 160

$$E(Y_{n+1} - \hat{\phi}_1 Y_n - \cdots - \hat{\phi}_p Y_{n+1-p})^2 \approx \sigma^2\left(1 + \frac{p}{n}\right)$$

Classical Decomposition $\quad X_t = m_t + s_t + Y_t \qquad EY_t = 0 \qquad s_{t+d} = s_t \qquad \sum_{j=1}^{d} s_j = 0$

No distortion $\Rightarrow$ $\quad m_t = \sum_j a_j m_{t-j}$ for all polynomials $m_t = c_0 + c_1 t + \cdots + c_k t^k$

No distortion $\Rightarrow$ $\quad \sum_j a_j = 1$ and $\sum_j j^r a_j = 0$ for $j = 1, \ldots, k$

Eliminate seasonal components $\Rightarrow$ $\quad \sum_j a_j s_{t-j} = \text{const} \times \sum_{j=1}^{d} s_j = 0$

Model	ACF	PACF
AR(p)	Decays	Zero for $h > p$
MA(q)	Zero for $h > q$	Decays
ARMA(p,q)	Decays	Decays